Now $\vec{L}\cdot\vec{S} = L_z S_z + \frac{1}{2}(L^+S^- + L^-S^+)$

where

$$\left.\begin{array}{l} L_z \\ L^+ = L_x + iL_y \\ L^- = L_x - iL_y \end{array}\right\} \text{ operates on } \phi_n(\vec{r})$$

and

$$\left.\begin{array}{l} S_z \\ S^+ = S_x + iS_y \\ S^- = S_x - iS_y \end{array}\right\} \text{ operates on } \alpha \text{ \& } \beta$$

with

$$S_z|\alpha\rangle = \frac{1}{2}|\alpha\rangle \quad , \quad S_z|\beta\rangle = -\frac{1}{2}|\beta\rangle$$

$$S^+|\beta\rangle = \alpha \quad , \quad S^+|\alpha\rangle = 0$$

$$S^-|\beta\rangle = 0 \quad , \quad S^-|\alpha\rangle = \beta$$

so that

$$\langle\alpha|S_z|\alpha\rangle = \frac{1}{2} \quad , \quad \langle\beta|S_z|\beta\rangle = -\frac{1}{2}$$

$$\langle\alpha|S^+|\beta\rangle = 1 \quad , \quad \langle\beta|S^-|\alpha\rangle = 1$$

others zero

C_n's

$$\underline{C_n^\alpha} = -\lambda \frac{\iint \phi_n^*(\vec{r})\alpha^*(\hat{L}_z\hat{S}_z)\,\phi_0(\vec{r})\,\alpha\, d\tau_s\, d\vec{r}}{E_n - E_0} - \frac{1}{2}\lambda \frac{\iint \phi_n^*(\vec{r})\beta^*(\hat{L}^+\hat{S}^-)\,\phi_0(\vec{r})\,\alpha\, d\tau_s\, d\vec{r}}{E_n - E_0}$$

$$= -\frac{\lambda}{2}\left[\frac{\int \phi_n^*(\vec{r})\,\hat{L}_z\,\phi_0(\vec{r})\,d\vec{r}}{E_n - E_0} \cdot \underbrace{\int \alpha^*\alpha\, d\tau_s}_{1} + \frac{\int \phi_n^*(\vec{r})\,\hat{L}^+\,\phi_0(\vec{r})\,d\vec{r}}{E_n - E_0} \cdot \underbrace{\int \beta^*\beta\, d\tau_s}_{1}\right]$$

$$\underline{C_n^\beta} = -\lambda \frac{\iint \phi_n^*(\vec r)\,\beta^*(\hat L_z \hat S_z)\,\phi_0(\vec r)\,\beta\, d\vec r\, d\tau_s}{E_n - E_0}$$

$$- \frac{\lambda}{2}\frac{\iint \phi_n^*(\vec r)\,\alpha^*(\hat L^- \hat S^+)\,\phi_0(\vec r)\,\beta\, d\vec r\, d\tau_s}{E_n - E_0}$$

$\int \beta^*\beta\, d\tau_s = 1$ $\quad$ $\int \alpha^*\alpha\, d\tau_s = 1$

$$= -\frac{\lambda}{2}\left[\frac{\int \phi_n^*(\vec r)\,\hat L_z\,\phi_0(\vec r)\,d\vec r}{E_n - E_0} + \frac{\int \phi_n^*(\vec r)\,\hat L^-\,\phi_0(\vec r)\,d\vec r}{E_n - E_0}\right]$$

Therefore

$$|+\rangle = \phi_0(\vec r)\cdot\alpha - \sum_n{}' \frac{\lambda}{2}\left(\frac{1}{E_n - E_0}\right)\left[\int \phi_n^*(\vec r)\,\hat L_z\,\phi_0(\vec r)\,d\tau + \int \phi_n^*(\vec r)\,\hat L^+\,\phi_0(\vec r)\,d\vec r\right]\phi_n(\vec r)$$

and

$$|-\rangle = \phi_0(\vec r)\cdot\beta - \sum_n{}' \frac{\lambda}{2}\left(\frac{1}{E_n - E_0}\right)\left[\int \phi_n^*(\vec r)\,\hat L_z\,\phi_0(\vec r)\,d\vec r + \int \phi_n^*(\vec r)\,\hat L^-\,\phi_0(\vec r)\,d\vec r\right]\phi_n(\vec r)$$

Zeeman Energy

$$\langle + |\, g_s\beta \vec S\cdot\vec H + g_L\beta \vec L\cdot\vec H \,| + \rangle$$

$$= \int \left(\phi_0^*(r)\,\alpha^* + \sum_n{}' c_n^*\,\psi_n^*(\vec r,\vec s)\right)\left(g_s\beta\,\vec S\cdot\vec H + g_L\beta\,\vec L\cdot\vec H\right)\left(\phi_0(\vec r)\,\alpha + \sum_n c_n\,\psi_n(\vec r,\vec s)\right)$$

$$= \int \phi_0^*(\vec r)\,\alpha^*\,(g_s\beta\,\vec S\cdot\vec H)\,\phi_0(\vec r)\,\alpha\, d\tau + \int \psi_0^*(\vec r,\alpha)\,(g_L\beta\,\vec L\cdot\vec H)\sum_n c_n\,\psi_n(\vec r,s)\,d\tau$$